ELEN E3401: Electromagnetics

Spring 2025

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Lecture #3

COLUMBIA ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Time and Phasor Domain

$$v(t) = \Re\left[\tilde{V} e^{j\omega t}\right]$$

$v(t)$		$\tilde{V}$
$A\cos(\omega t)$	$\leftrightarrow$	A
$A\cos(\omega t + \phi)$	$\leftrightarrow$	$Ae^{j\phi}$
$-A\cos(\omega t + \phi)$	$\leftrightarrow$	$Ae^{j(\phi \pm \pi)}$
$A\sin(\omega t)$	$\leftrightarrow$	$Ae^{-j\pi/2} = -jA$
$A\sin(\omega t + \phi)$	$\leftrightarrow$	$Ae^{j(\phi - \pi/2)}$
$-A\sin(\omega t + \phi)$	$\leftrightarrow$	$Ae^{j(\phi + \pi/2)}$
$A\cos(\omega t + \beta z + \phi_0)$	$\leftrightarrow$	$Ae^{j(\beta z + \phi_0)}$
$Ae^{-\alpha z}\cos(\omega t + \beta z + \phi_0)$	$\leftrightarrow$	$Ae^{-\alpha z}e^{j(\beta z + \phi_0)}$

Traveling waves in Phasor Domain

Much easier to deal with multiplying phasor domain exponentials than time domain integral-differential equation

Time and Phasor Domain

$$v(t) = \Re\left[\tilde{V} e^{j\omega t}\right]$$

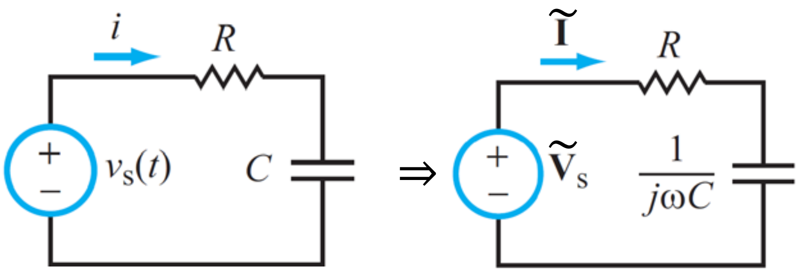
$v(t)$		$\tilde{V}$
$\dfrac{d}{dt}\big(v(t)\big)$	$\leftrightarrow$	$j\omega\tilde{V}$
$\dfrac{d}{dt}(A\cos(\omega t + \phi))$	$\leftrightarrow$	$j\omega A e^{j\phi}$
$\int v(t)dt$	$\leftrightarrow$	$(1/j\omega)\tilde{V}$
$\int A\cos(\omega t + \phi)\,dt$	$\leftrightarrow$	$(1/j\omega)Ae^{j\phi}$
$\int A\sin(\omega t + \phi)\,dt$	$\leftrightarrow$	$(1/j\omega)Ae^{j(\phi_0 - \pi/2)}$

ac Phasor Analysis: General Procedure

$$v_s(t) = V_0 \sin(\omega t + \phi_0)$$
$$= V_0 \cos\left(\omega t + \phi_0 - \frac{\pi}{2}\right)$$
$$= \Re\left[V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} e^{j\omega t}\right]$$
$$\Rightarrow \tilde{V}_s = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)}$$

1. Adopt Cosine Reference for Source

2. Transfer to Time-Independent Phasor Domain



$$i \rightarrow \tilde{I}$$
$$v \rightarrow \tilde{V}$$
$$R \rightarrow \tilde{Z}_R = R$$
$$L \rightarrow \tilde{Z}_L = j\omega L$$
$$C \rightarrow \tilde{Z}_C = 1/j\omega C$$

<u>Apply Kirchoff's Voltage Law (KVL)</u>

Time Domain: $Ri(t) + \frac{1}{C}\int i(t)\partial t = v_s(t)$

Phasor Domain: $\tilde{I}\left(\tilde{Z}_R + \tilde{Z}_C\right) = \tilde{I}\left(R + \frac{1}{j\omega C}\right) = \tilde{V}_s$

3. Obtain Phasor Form for Time Domain Equation

$$\tilde{I} = \frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} = I_0 e^{j\phi}$$

4. Solve Phasor Domain Equation for unknown Variable ($\tilde{I}$)

ac Phasor Analysis: General Procedure

4. Solve Phasor Domain Equation (cont')

Recall: $j = e^{j\pi/2}$

$$\tilde{I} = \frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} \Rightarrow \tilde{I} = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} \frac{1}{R + \frac{1}{j\omega C}} = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} \frac{j\omega C}{1 + j\omega RC} = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} \frac{e^{j\frac{\pi}{2}}\omega C}{1 + j\omega RC}$$

Recall: $x + jy \leftrightarrow |z|e^{j\phi}$

Denominator: $1 + j\omega RC = \sqrt{1 + \omega^2 R^2 C^2} e^{j\phi_1}, \quad \phi_1 = \tan^{-1}(\omega RC)$

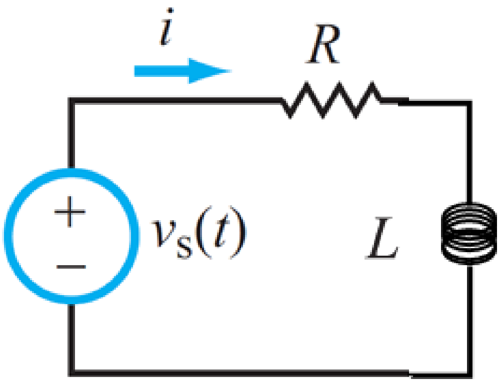
$$\tilde{I} = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} \frac{e^{j\frac{\pi}{2}}\omega C}{\sqrt{1 + \omega^2 R^2 C^2} e^{j\phi_1}} = \frac{V_0 \omega C}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j(\phi_0 - \phi_1)}$$

5. Transform Back to Time Domain

$$i(t) = \Re\left[\tilde{I} e^{j\omega t}\right] = \Re\left[\frac{V_0 \omega C}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j(\phi_0 - \phi_1)} e^{j\omega t}\right] = \frac{V_0 \omega C}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t + \phi_0 - \phi_1)$$

$$\boxed{i(t) = \frac{V_0 \omega C}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t + \phi_0 - \phi_1), \quad \phi_1 = \tan^{-1}(\omega RC)}$$

RL Circuit Phasor Example



Given $v_s(t)$
Find $i(t)$

Consider: $v_s(t) = 150\cos(\omega t)$

$$R = 400[\Omega]$$
$$L = 3[mH]$$
$$\omega = 10^5\left[\frac{rad}{s}\right]$$

<u>KVL</u>

Time Domain: $Ri(t) + L\frac{\partial i}{\partial t} = v_s(t)$

Phasor Domain: $R\tilde{I} + j\omega L\tilde{I} = \tilde{V}_s$

<u>Solve for</u> $\tilde{I} = \frac{\tilde{V}_s}{R+j\omega L}$ $\quad \tilde{V}_s = 150\angle 0^o$

RL Circuit Phasor Example

$$\tilde{I} = \frac{150}{400 + j(10^5)(3 \times 10^{-3})} = \frac{150}{400 + j300}$$

$$\tan^{-1}(300/400) \ = 36.9^o = 0.6435 \ [rad]$$

$$\tilde{I} = \frac{150}{\sqrt{400^2 + 300^2} e^{j36.9^o}} = \frac{150}{500 e^{j36.9^o}}$$

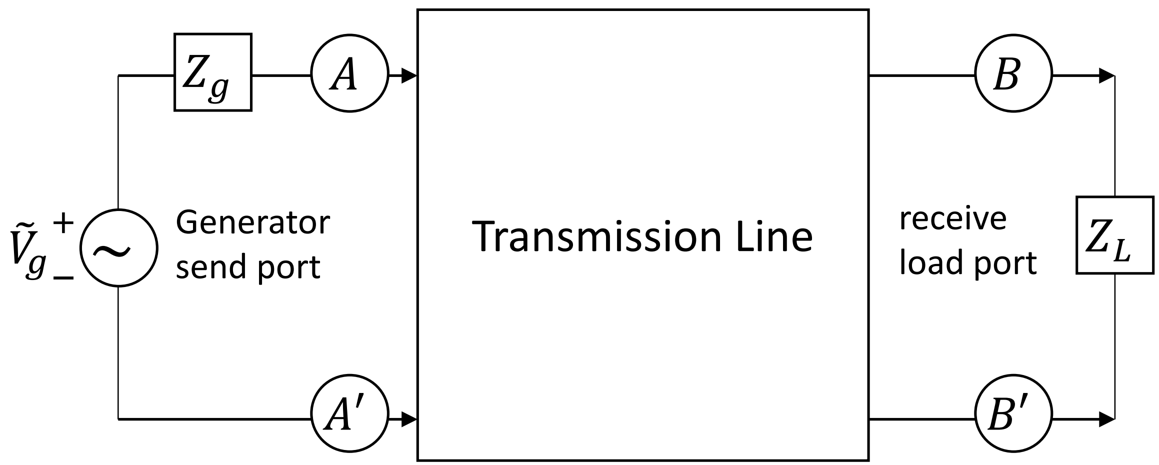
$$\tilde{I} = \frac{150}{500} e^{-j36.9^o} = 0.3\angle - 36.9^o$$

$$i(t) = \Re\left[\tilde{I} e^{j\omega t}\right] = \Re\left[0.3 e^{-j36.9^o} e^{j10^5 t}\right]$$

$$i(t) = 0.3\cos(10^5 t - 36.9^o)$$

Transmission Line

- 2-port network



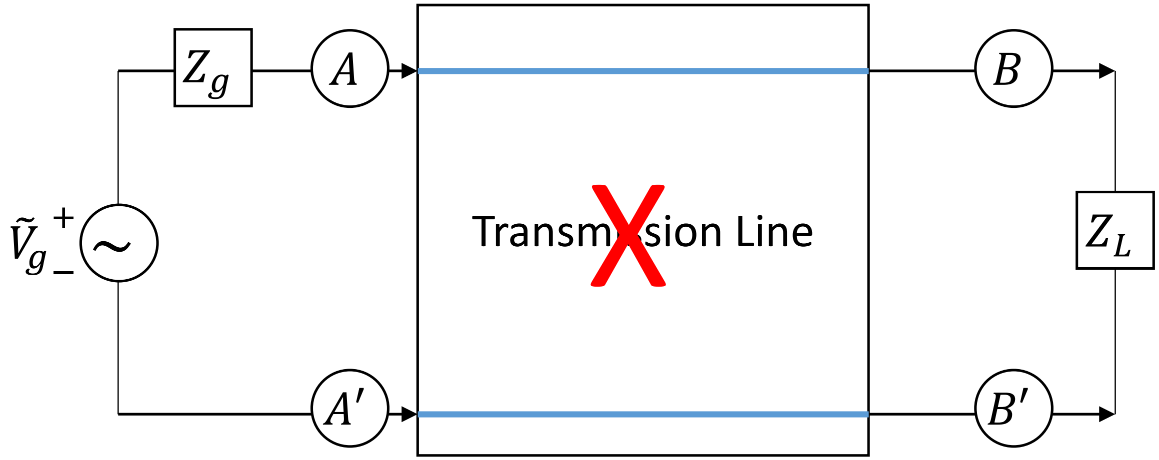
$\tilde{V}_g$ = source ac signal $[V]$

Z_g = source impedance $[\Omega]$

Z_L = load impedance $[\Omega]$

- Thévenin Equivalent generator circuit
- Generator voltage Vg in series with generator resistance, Rg
- For AC signals → $\tilde{V}_g$ and Z_g impedance
- Load impedance → Z_L

Transmission Line



$\tilde{V}_g$ = source ac signal $[V]$

Z_g = source impedance $[\Omega]$

Z_L = load impedance $[\Omega]$

- When does a simple 2-wires circuit need to be treated as a transmission line?
- Factors = wire length and signal frequency

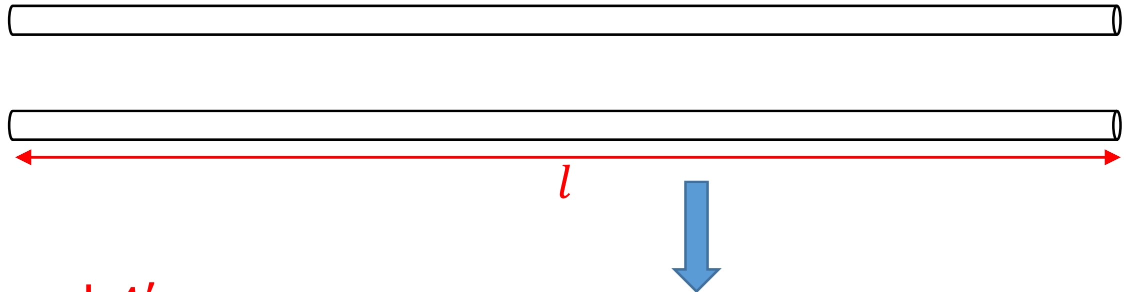
Key ratio is $\frac{l}{\lambda}$: (length of the line and signal wavelength)

$\frac{l}{\lambda} \ll 1 \Rightarrow$ can ignore transmission line

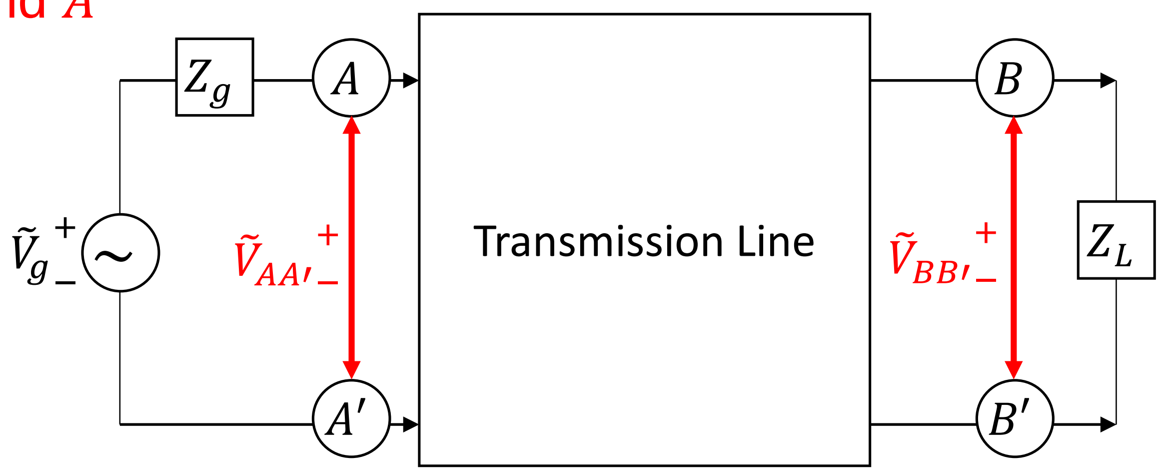
$\frac{l}{\lambda} \gtrsim 0.01 \Rightarrow$ must consider transmission line

Transmission line

Pair of wires as transmission line



Consider: voltage across A and A'



$$V_{AA'} = V_g(t) = V_0 \cos(\omega t)$$

$$\omega = 2\pi f$$

Assume current flowing at:

$$u_p = c \approx 3 \times 10^8 \left[\frac{m}{s}\right]$$

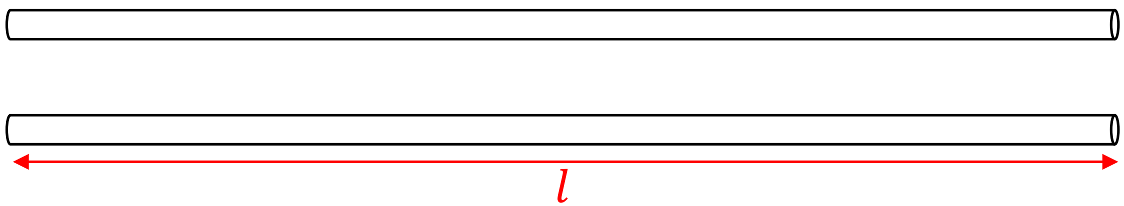
Voltage across B and B' delayed by: ${}^{l}/_{c}$

where l = length of wire

$$V_{BB'} = V_{AA'}\left(t - {}^{l}/_{c}\right)$$
$$= V_0 \cos\left(\omega\left(t - {}^{l}/_{c}\right)\right)$$
$$= V_0 \cos(\omega t - \phi_0)$$
$$\Longrightarrow \phi_0 = {}^{\omega l}/_{c}$$

Time delay ⟺ Phase shift

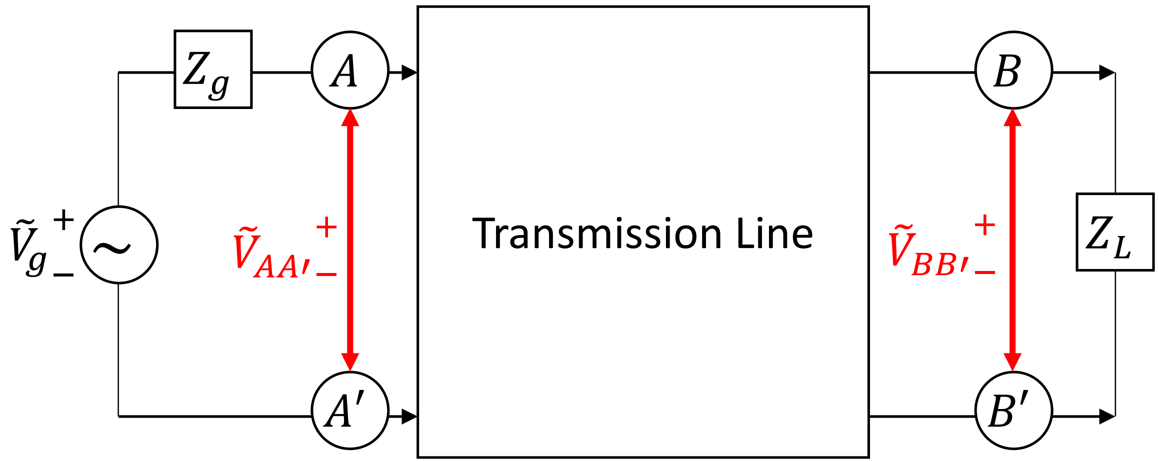
Transmission Line Length vs Wavelength



Set $t = 0$

<u>Consider:</u>

- $f = 1\ [kHz]$
- $l = 5\ [cm]$
- $V_{AA'} = V_0$

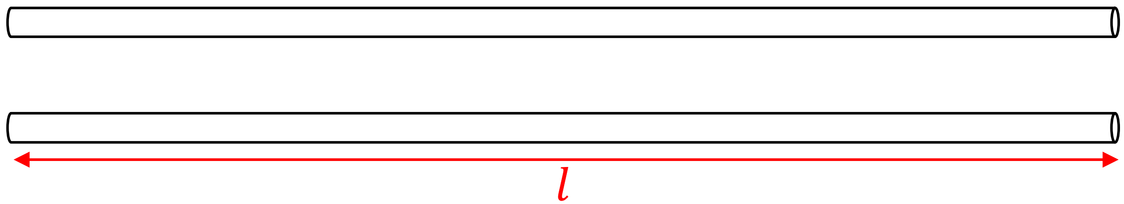


$$V_{BB'} = V_0 \cos\left(2\pi f\, {}^{l}/_{c}\right)$$
$$= 0.9999999998 \times V_0$$
$$\boldsymbol{V_{AA'} \approx V_{BB'}}$$

Now…if we increase f to 10 GHz and $l = 10cm$

$$V_{BB'} = V_0 \cos\left(2\pi f\, {}^{l}/_{c}\right) = 0.93 \times V_0$$
$$\boldsymbol{V_{AA'} \neq V_{BB'}} \leftarrow \text{significant!}$$

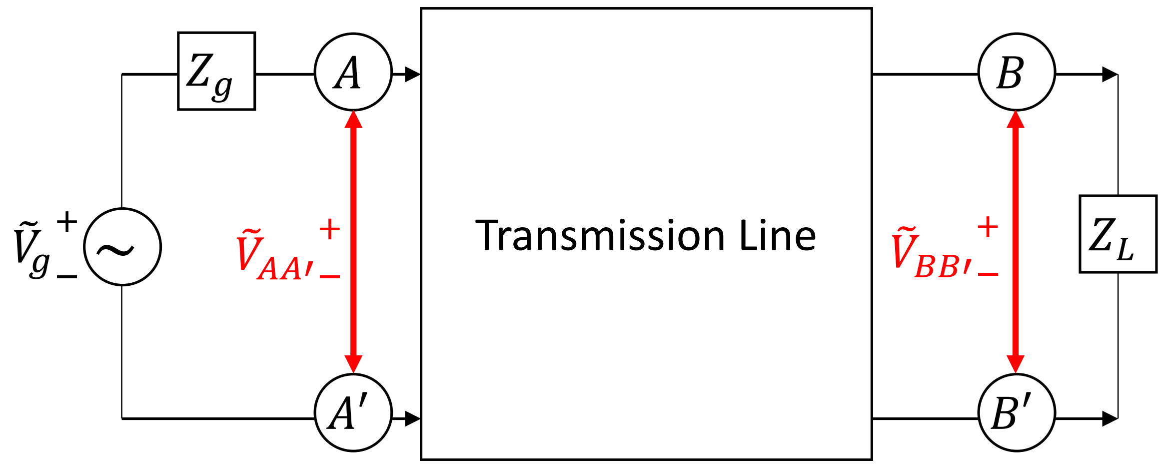
Transmission Line Length vs Wavelength



$\phi_0 = {}^{\omega l}/_{c}$ & $u_p = f\lambda$

if $u_p = c$

$\Longrightarrow \boldsymbol{\phi_0} = {}^{2\pi f l}/_{c} = {}^{\boldsymbol{2\pi l}}/_{\boldsymbol{\lambda}}$
(phase delay)

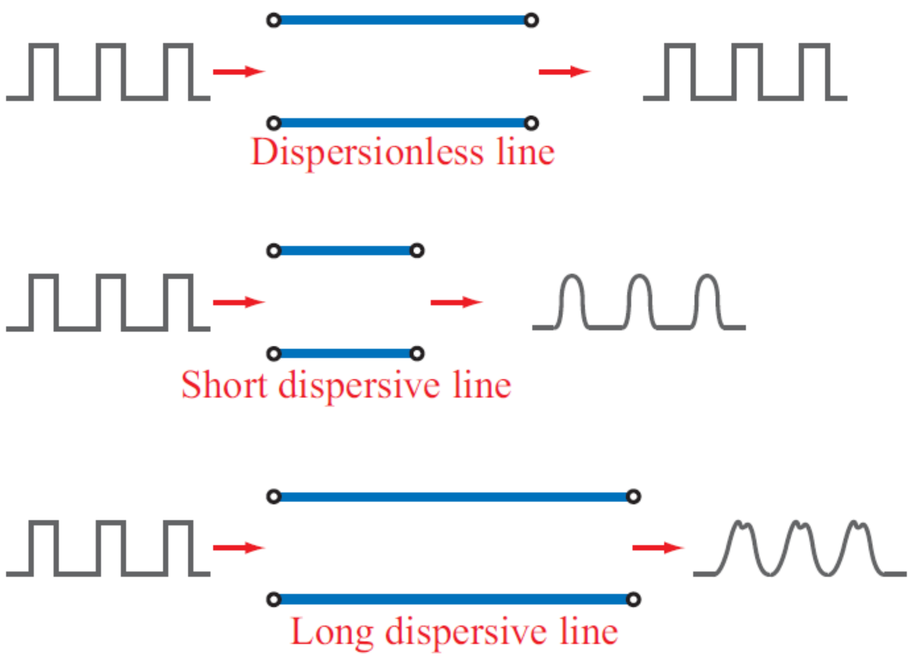


$$if \quad \frac{l}{\lambda} \gtrsim 0.01$$

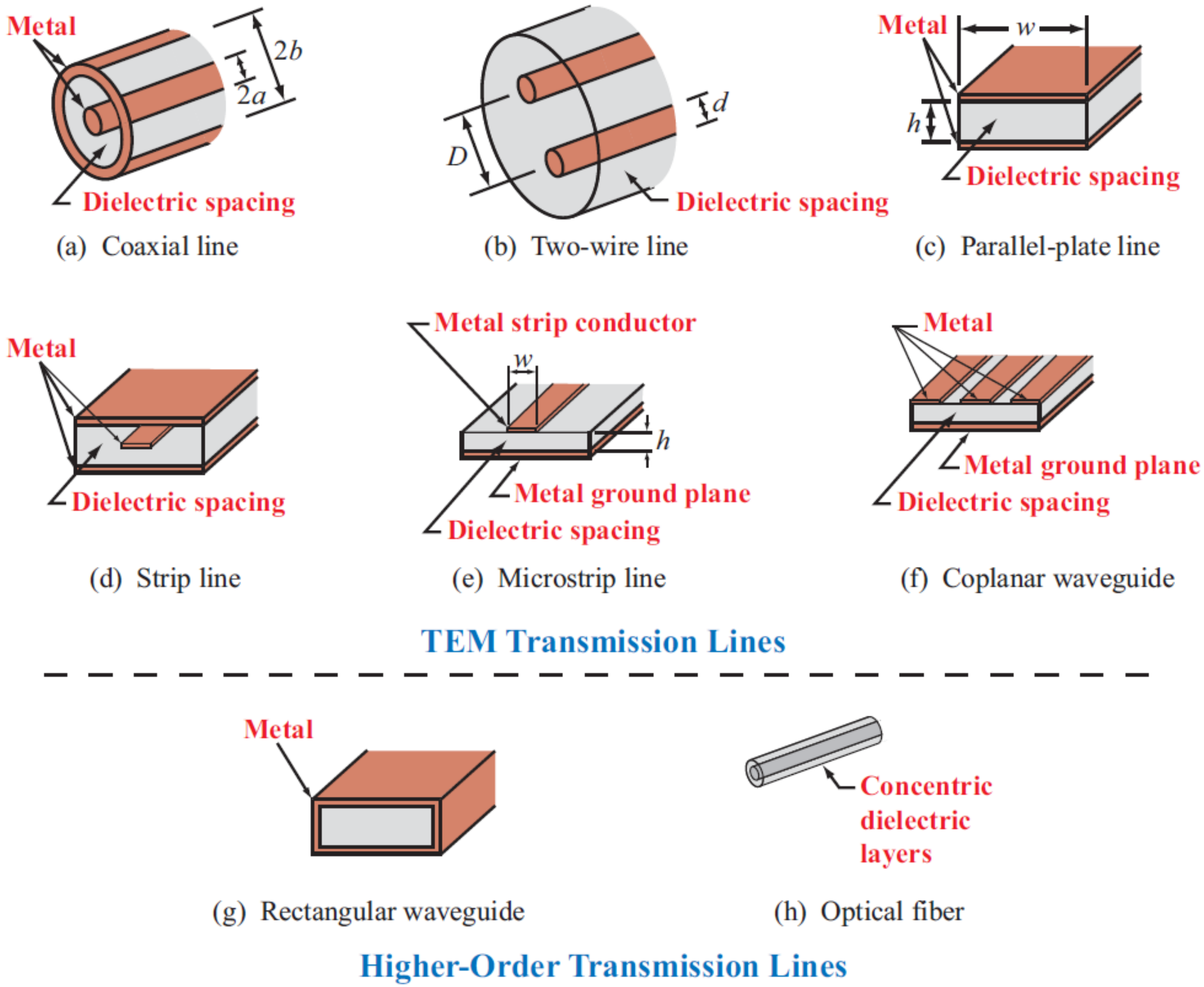
$\Rightarrow$ must consider transmission line phase shift
<u>and</u> reflected signals from load to generator

Dispersive Line

- Transmission line where wave (phase) velocity is a function of frequency
- Dispersive line will distort signal shape.
- At high frequencies, (~10GHz) even short (mm) wires can be dispersive and the transmission line must be considered.



Types of Transmission Lines



Types of transmission line

- Parameters will have different expressions that depend on geometry and EM constitutive parameters of the TEM line.

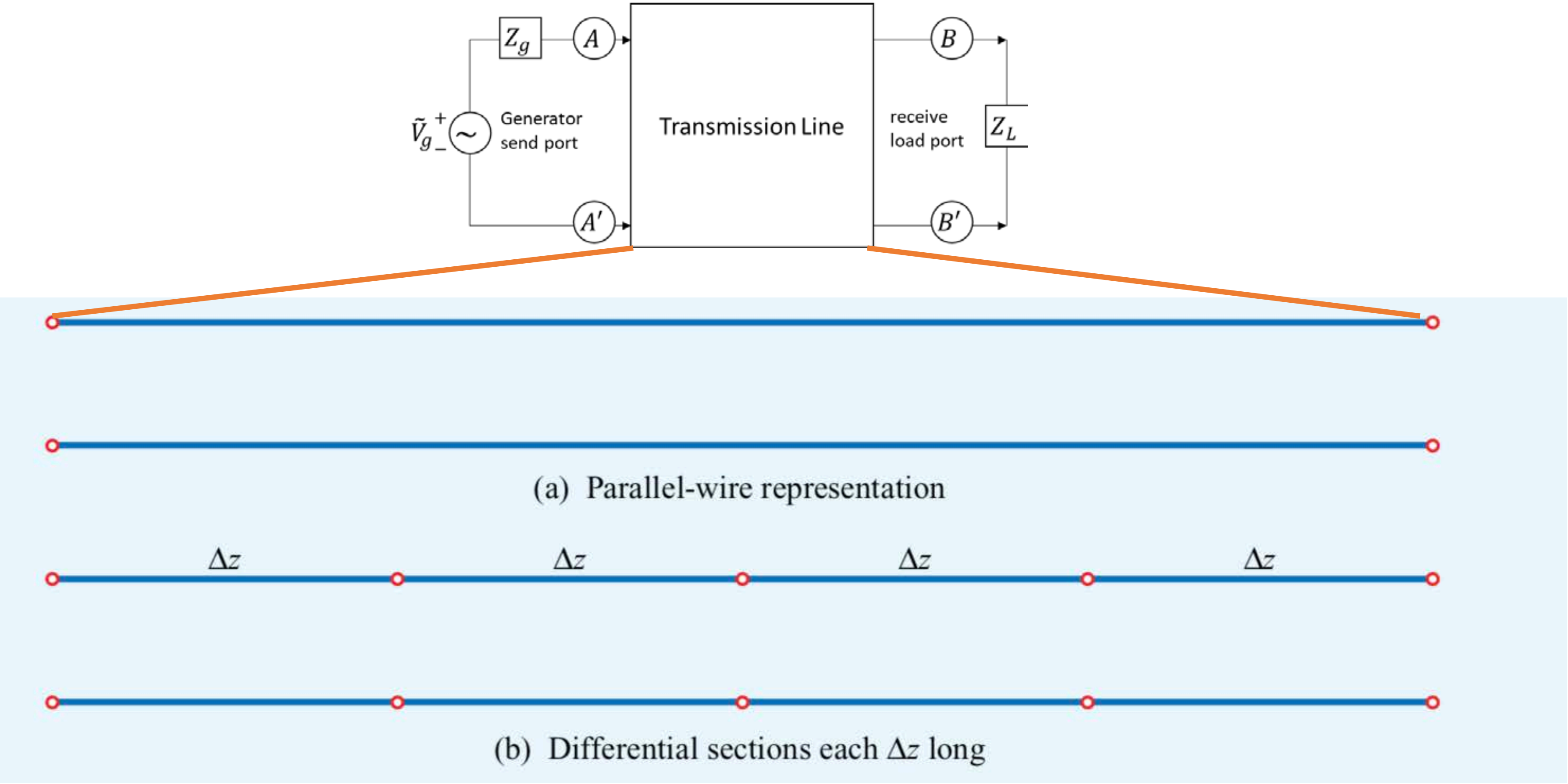
1. Coax line
   - a = outer radius of inner conductor
   - b = inner radius of outer conductor

2. Two wire line
   - d = diameter of each wire
   - D = spacing between wires

3. Parallel-plate line
   - w = width of each plate
   - h = thickness of insulation between plates

Lumped-Element Model

- Model transmission line as parallel-wire with lumped elements.
- Regardless of actual transmission line shape



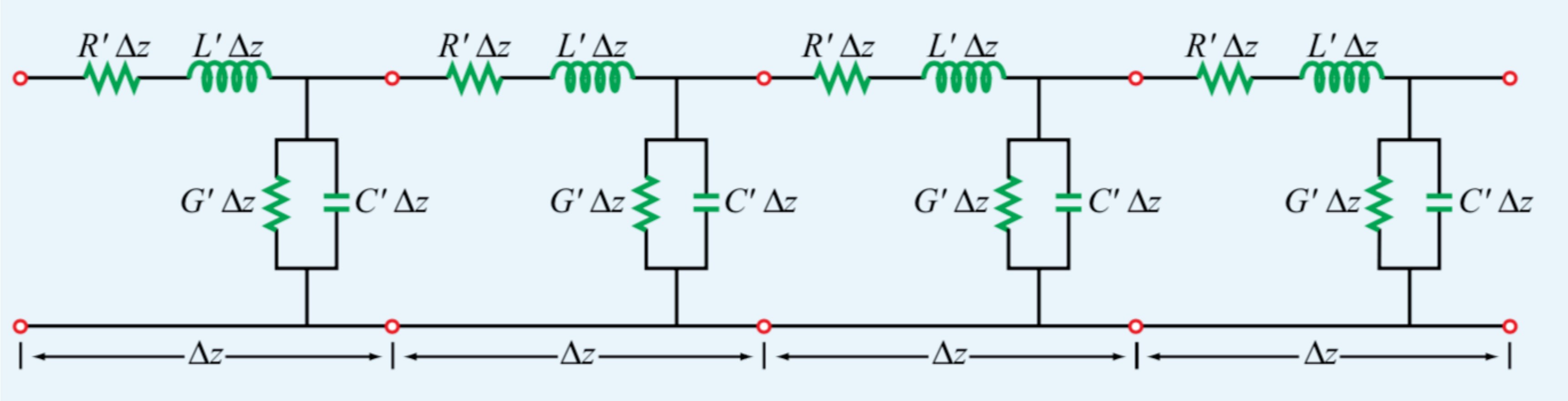
(a) Parallel-wire representation

(b) Differential sections each Δz long

Lumped-Element Model

Each section represented by equivalent circuit:

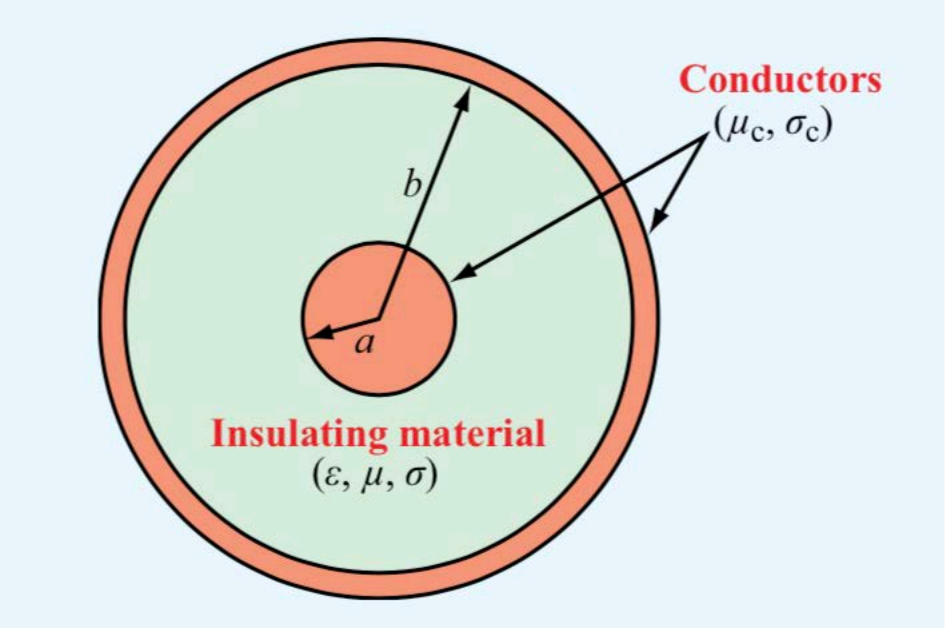
(b) Differential sections each Δz long



Transmission Line Parameters:

1. R’ = combined resistance of both conductors per unit length, Ω/m
2. L’ = combined inductance of both conductors per unit length, H/m
3. G’ = conductance of insulation between two conductors per unit length, S/m
4. C’ = capacitance of two conductors per unit length, F/m

Example: Coax Line



$$R' = \frac{R_S}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \text{ [}\Omega\text{/m]}$$

$$L' = \frac{\mu}{2\pi} ln\left(\frac{b}{a}\right) \text{ [H/m]}$$

$$G' = \frac{2\pi\sigma}{ln\left(\frac{b}{a}\right)} \text{ [S/m]}$$

$$C' = \frac{2\pi\epsilon}{ln\left(\frac{b}{a}\right)} \text{ [F/m]}$$

Voltage applied across terminals at two conductors, current flows along outer surface of inner conductor and inner surface of outer conductor

Coax Line

We will derive these expressions later in the course

Line resistance accounts for both conductors:

$$R' = \frac{R_S}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \text{ [}\Omega\text{/m]}$$

$$R_s = surface\ resistance = \sqrt{\frac{\pi f \mu_c}{\sigma_c}} \text{ [}\Omega\text{]}$$

Perfect conductor : $\sigma_c = \infty$

$\frac{f\mu_c}{\sigma_c} << 1$, $R_s \to 0$, $R' \to 0$

Line inductance, L', accounts for joint inductance of two conductors

$$L' = \frac{\mu}{2\pi} ln\left(\frac{b}{a}\right) \text{ [H/m]}$$

We will show this later in the course with Ampere's Law

Coax Line

- Line conductance, G', accounts for current flow between outer/inner conductors.
- Current flows from one conductor to the second. Possible through insulator.
- G' → shunt element

$$G' = \frac{2\pi\sigma}{ln\left(\frac{b}{a}\right)} \text{ [S/m]},$$

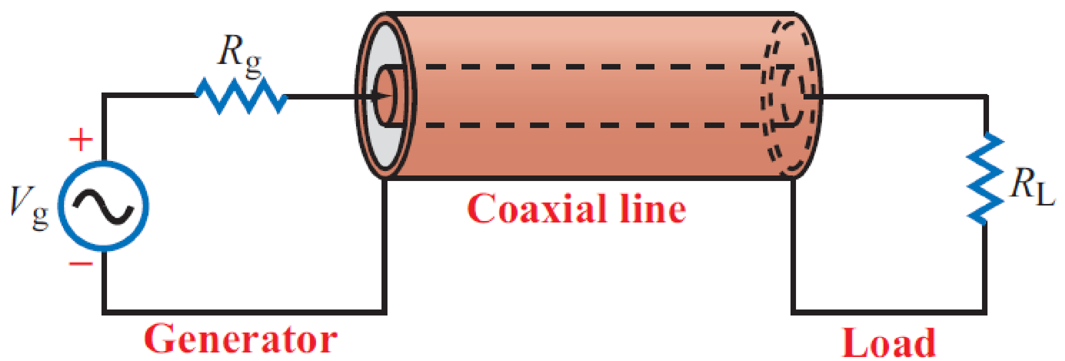
- where σ is the insulator conductivity
- If material separating inner/outer conductors is a <u>perfect dielectric, σ = 0, then G' = 0</u>

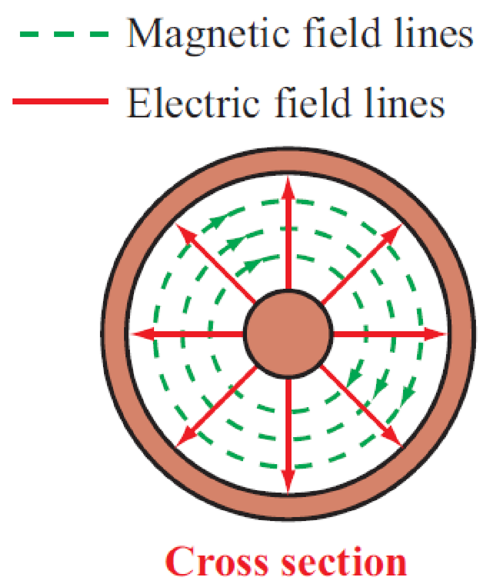
- Line capacitance, C', develops from equal/opposite charge on non-contacting conductors, leads to voltage drop.

$$\text{Capacitance} = \frac{charge}{voltage\ drop}$$

$$C' = \frac{2\pi\epsilon}{ln\left(\frac{b}{a}\right)} \text{ [F/m]}$$

TEM Mode in Coax transmission line





Electric Field E is radial
Magnetic Field H is azimuthal
Propagation is into the page